On the Length of the Day depending on Latitude and Date

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Abstract

We derive a formula to calculate the length of the day at a given latitude at a date of year.

1 Introduction

Sometimes one wonders how does the length of the day change during the year at some given location. In this short paper we derive the formula that answers the question.

We first define some variables. We states the task in two coordinate systems.

Three solutions are given: one on the polar circle, one below - with the special case of the inclination of the Earth and of a Location with the latitude of Berne - and one above.

2 Latitude and Date of the Year

Usually the latitude is measured in degree from the Equator to the Poles. We follow the opposite direction and we use radiants.

The poles have latitude in radiant 0 (the north pole) and π (the south pole.)

The equator has latitude $\pi/2$.

We denote with ω the latitude of the polar circle. The formulas are valid for $0 \leq \omega < \frac{\pi}{2}$ (we ignore the singularity when the Nort Pole points to the Sun).

We denote with λ the latitude for which we want to calculate the length of the day, (when $\lambda > \omega$ the location is below the polar circle.)

The date of the year is measured in radiant ($1year = 2\pi$) starting from the summer solstice (on Earth about the 21 June) and we denote it with t . We shall also measure the length of the day in radiant (i.e. $24^h = 2\pi$).

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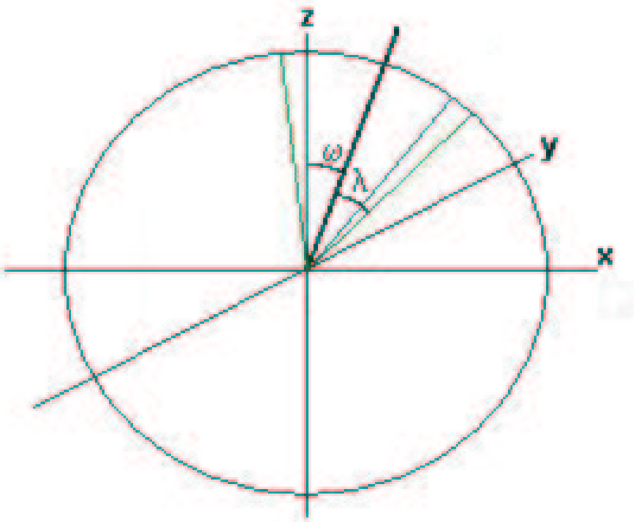
3 The First Coordinate System



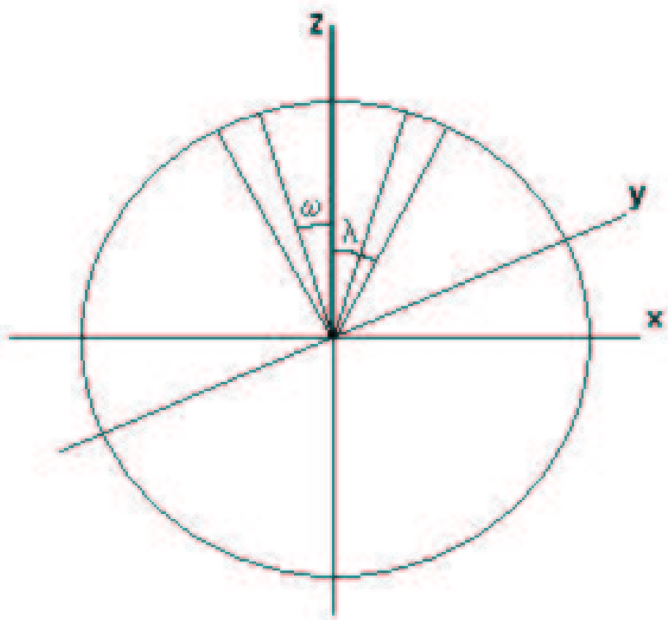
We choose the following coordinate system:

1. The rotation axis of the Earth is fixed and lies in the plane $y = 0$.
2. The centre of the Earth is the origin of the coordinate system.
3. At the time zero the Sun is at infinity on the positive x-Axis.
4. The Sun moves on the $z = 0$ plane.
5. The diameter of the Earth is 1.

In the coordinate system the follow is true:

1. The Earth is the unit sphere with centre the origin.
2. A unit vector that from the origin points to the center of the Sun $\vec{n}_t$ satisfies the following equation: $\vec{n}_t = (cos(t), sin(t), 0)$.
3. The day zone and the night zone intersect on a grand circle of the unit sphere. This circle determines a plane through the origin $P_t : \{(X, Y, Z) \in R^3 \mid \vec{n}_t.(X, Y, Z) = 0\}$.
4. The angle between the z-axis and the North Pole is constant and equal to ω. The North Pole N is fixed with coordinates $N = (sin(\omega), 0, cos(\omega))$.
5. As the earth turns on itself the point $(0, 0, 1)$ draws the polar circle on the surface of the unit sphere. We denotes the circle with C_ω and the cone with edge the origin and mantelline the circle C_ω with K_ω .
6. In a similar way the parallel of the location is denoted with C_λ and its cone with K_λ. The location cone $K_\lambda$contains the polar cone K_ω.

4 The New Coordinate System



We rotate around the y-axis of the amount ω (now the coordinates of the North Pole are $(0,0,1)$).

In the old system the three unit vectors of the coordinate system where:

$$\begin{aligned}\vec{x} &= (1,0,0) = \vec{n}_0 \\ \vec{y} &= (0,1,0) \\ \vec{z} &= (0,0,1)\end{aligned}$$

Note also that $\vec{n}_t = \vec{x}\cos t + \vec{y}\sin t$.

In the new system they are now

$$\begin{aligned}\vec{x}' &= (\cos\omega, 0, \sin\omega) \\ \vec{y}' &= (0,1,0) \\ \vec{z}' &= (-\sin\omega, 0, \cos\omega)\end{aligned}$$

The orthogonal vector to the plane P_t (i.e. the direction to the Sun) is

$$\begin{aligned}\vec{n}'_t &= \vec{x}'\cos t + \vec{y}'\sin t \\ &= (\cos t\cos\omega, \sin t, \cos t\sin\omega)\end{aligned}$$

That gives us the equation of the plane P_t

$$0 = \vec{n}' \cdot (x, y, z)$$

In order to determine the length of the day we must find the intersection between the plane P_t and the location parallel C_λ. The equation of the location parallel is:

$$\begin{aligned}z &= \cos\lambda \\ x^2 + y^2 &= \sin^2\lambda\end{aligned}$$

The two equations above must be solved together with the equation of the plane P_t . So the problem after having substituted z with its value $\cos\lambda$ is to solve the two equations with variables x and y for the three parameters: the time t , the inclination of the Earth ω and the latitude of the location λ :

$$\begin{aligned}x^2 + y^2 &= \sin^2\lambda \\ \cos t\cos\omega\cdot x + \sin t\cdot y &= -\cos t\sin\omega\cos\lambda\end{aligned}$$

The system may have zero, one or two solutions. In the first two cases the day has either length *zero* or 2π in the last case the length of the day is the angle between the two vectors: (x_0, y_0) and (x_1, y_1).

The computation (which was performed with **maxima** - cf. appendix) returns the following solutions:

$$\begin{aligned}x_\pm &= -\frac{\cos\lambda\cos^2 t\cos\omega\sin\omega \pm \sin t\sqrt{\sin^2\lambda\sin^2 t + \sin^2\lambda\cos^2 t\cos^2\omega - \cos^2\lambda\cos^2 t\sin^2\omega}}{\cos^2 t\cos^2\omega + \sin^2 t} \\ y_\pm &= \frac{-\cos\lambda\cos t\sin t\sin\omega \pm \cos t\cos\omega\sqrt{\sin^2\lambda\sin^2 t + \sin^2\lambda\cos^2 t\cos^2\omega - \cos^2\lambda\cos^2 t\sin^2\omega}}{\cos^2 t\cos^2\omega + \sin^2 t}\end{aligned}$$

We can simplify the denominator :

$$\begin{aligned}
\cos^2 t \cos^2 \omega + \sin^2 t &= \cos^2 t \cos^2 \omega - \cos^2 t + 1 \\
&= \cos^2 t \cdot (\cos^2 \omega - 1) + 1 \\
&= 1 - \cos^2 t \sin^2 \omega
\end{aligned}$$

Also the term in the square root can be simplified

$$\begin{aligned}
sqrtexpression &= \sin^2 \lambda \sin^2 t + \sin^2 \lambda \cos^2 t \cos^2 \omega - \cos^2 \lambda \cos^2 t \sin^2 \omega \\
&= \sin^2 \lambda (1 - \cos^2 t) + \sin^2 \lambda \cos^2 t \cos^2 \omega - \cos^2 \lambda \cos^2 t \sin^2 \omega \\
&= \sin^2 \lambda - \sin^2 \lambda \cos^2 t + \sin^2 \lambda \cos^2 t \cos^2 \omega - \cos^2 \lambda \cos^2 t \sin^2 \omega \\
&= \sin^2 \lambda - \sin^2 \lambda \cos^2 t \cdot (1 - \cos^2 \omega) - \cos^2 \lambda \cos^2 t \sin^2 \omega \\
&= \sin^2 \lambda - \sin^2 \lambda \cos^2 t \cdot \sin^2 \omega - \cos^2 \lambda \cos^2 t \sin^2 \omega \\
&= \sin^2 \lambda - (\sin^2 \lambda + \cos^2 \lambda) \cdot \cos^2 t \cdot \sin^2 \omega \\
&= \sin^2 \lambda - \cos^2 t \sin^2 \omega
\end{aligned}$$

With the simplifications the solutions are now:

$$\begin{aligned}
x_\pm &= -\frac{\cos \lambda \cos^2 t \cos \omega \sin \omega \pm \sin t \sqrt{\sin^2 \lambda - \cos^2 t \cdot \sin^2 \omega}}{1 - \cos^2 t \sin^2 \omega} \\
y_\pm &= \frac{-\cos \lambda \cos t \sin t \sin \omega \pm \cos t \cos \omega \sqrt{\sin^2 \lambda - \cos^2 t \cdot \sin^2 \omega}}{1 - \cos^2 t \sin^2 \omega}
\end{aligned}$$

4.1 The Case of the Polar Circle

When $\lambda = \omega$ (i.e. the location lies on the polar circle) the term under the square root becomes

$$\sin^2 \omega - \cos^2 t \cdot \sin^2 \omega = \sin^2 \omega \cdot (1 - \cos^2 t) = \sin^2 \omega \cdot \sin^2 t = (\sin \omega \cdot \sin t)^2$$

and the equations are now

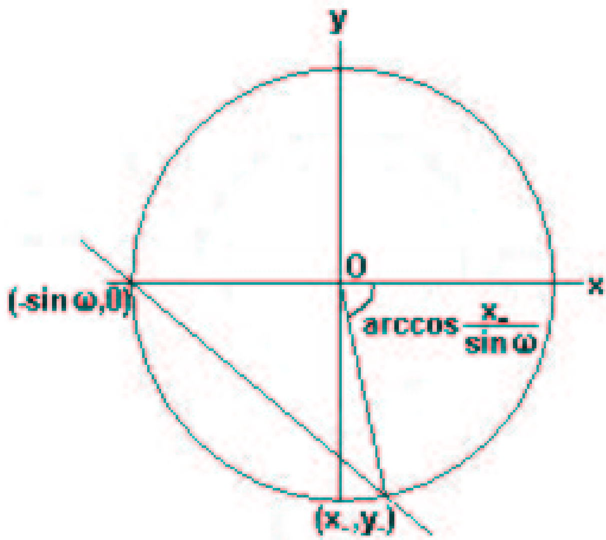
$$\begin{aligned}
x_\pm &= -\frac{\cos^2 t \cos^2 \omega \sin \omega \pm \sin^2 t \sin \omega}{1 - \cos^2 t \sin^2 \omega} \\
&= -\sin \omega \frac{\cos^2 t \cos^2 \omega \pm \sin^2 t}{1 - \cos^2 t \sin^2 \omega} \\
y_\pm &= \frac{-\cos \omega \cos t \sin t \sin \omega \pm \cos t \cos \omega \sin \omega \sin t}{1 - \cos^2 t \sin^2 \omega} \\
&= \cos \omega \cos t \sin t \sin \omega \frac{-1 \pm 1}{1 - \cos^2 t \sin^2 \omega}
\end{aligned}$$

Let expand the elements of the solutions

$$\begin{aligned}
x_+ &= -\sin \omega \frac{\cos^2 t \cos^2 \omega + (1 - \cos^2 t)}{1 - \cos^2 t \sin^2 \omega} \\
&= -\sin \omega \frac{\cos^2 t \cdot (\cos^2 \omega - 1) + 1}{1 - \cos^2 t \sin^2 \omega}
\end{aligned}$$

$$
\begin{aligned}
&= -\sin\omega \frac{\cos^2 t \cdot (-\sin^2\omega) + 1}{1 - \cos^2 t \sin^2\omega} \\
&= -\sin\omega \\
y_+ &= 0 \\
x_- &= \sin\omega \frac{\sin^2 t - \cos^2 t \cos^2\omega}{1 - \cos^2 t \sin^2\omega} \\
y_- &= \sin\omega \frac{-2\cos\omega \cos t \sin t}{1 - \cos^2 t \sin^2\omega}
\end{aligned}
$$

4.1.1 The Length of the Day on the Polar Circle

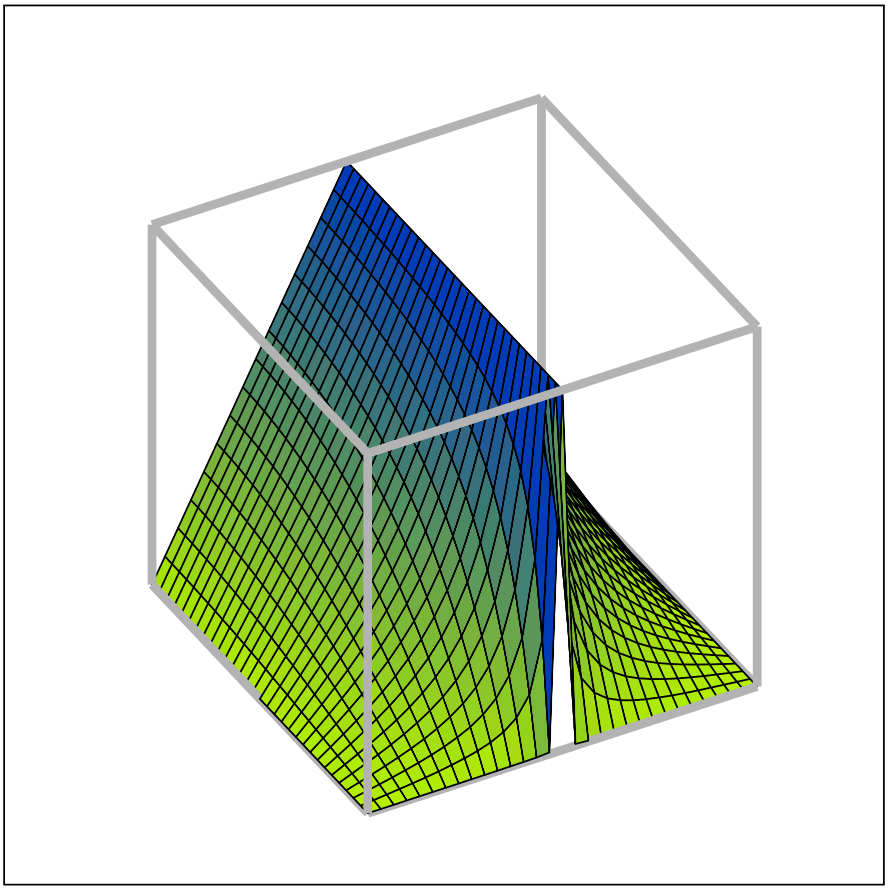


From a simple drawing it is clear that for$-\frac{\pi}{2} < t < \frac{\pi}{2}$the duration of the day d_t satisfies the equation:

$$\begin{aligned} d_t &= \frac{\pi}{2} + \arccos \frac{x_-}{\sin\omega} \\ &= \frac{\pi}{2} + \arccos \frac{\sin^2 t - \cos^2 t \cos^2 \omega}{1 - \cos^2 t \sin^2 \omega} \end{aligned}$$

Graphically we have (the origin is on the middle of the edge behind the surface and on the left there is the $t - axis$ and on the right the $\omega - axis$) the following picture. Note that the wedge in front has coordinates $(t, \omega\, d) = (\frac{\pi}{2}, \frac{\pi}{2}, 2\pi)$ and the bottom plane has height $d = \pi$.

Plot of z =



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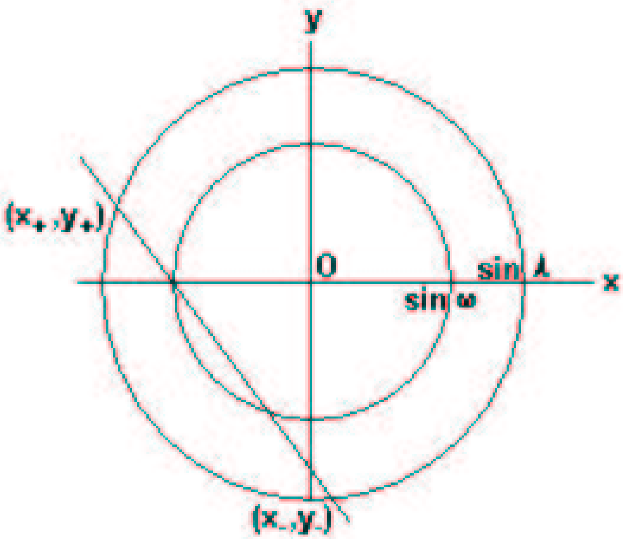
We recall that $t = 0$ corresponds to the solstice of Summer where for each latitude of the polar circle (i.e. inclination of the Earth rotaion axis) the day lasts π (i.e. 12^h).On the other hand for the autumnal equinox $t = \frac{\pi}{2}$ or for the spring equinox $t = -\frac{\pi}{2}$ when everywhere the day lasts π (i.e. 12^h).

The graphic can be continued in the ω direction by axial symmetry around the axis $t = \frac{\pi}{2}$ (and $d = \pi$).

It is interesting to see that almost without inclination, the day gradually (linearly) become shorter as time passes but with increasing inclination the day stay almost the whole year at about 12^h and suddenly increases to 24^h (or drop to 0^h) has the solstice approaches.

4.2 Below the Polar Circle



We have $\lambda > \omega$ and the length of the day is now $d_t = \arccos \frac{x_+}{\sin \lambda} + \arccos \frac{x_-}{\sin \lambda}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

In the full form:

$$\begin{aligned} d_t &= \arccos - \frac{\cos \lambda \cos^2 t \cos \omega \sin \omega + \sin t \sqrt{\sin^2 \lambda - \cos^2 t \cdot \sin^2 \omega}}{\sin \lambda \cdot (1 - \cos^2 t \sin^2 \omega)} \\ &+ \arccos - \frac{\cos \lambda \cos^2 t \cos \omega \sin \omega - \sin t \sqrt{\sin^2 \lambda - \cos^2 t \cdot \sin^2 \omega}}{\sin \lambda \cdot (1 - \cos^2 t \sin^2 \omega)} \end{aligned}$$

On Earth the latitude ω_0 of the Polar Circle is about $23^o 27'$. Numerically we have:

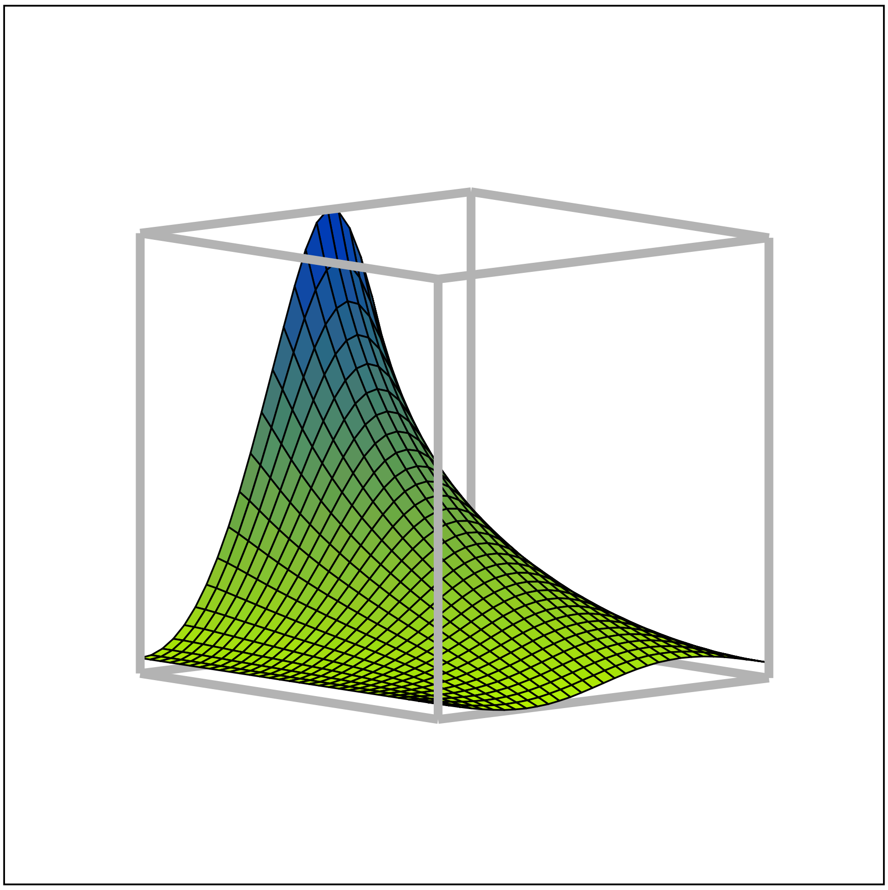
$$\begin{aligned} &\sin \omega_0 \approx 0.3979.. \approx 0.3846... \approx \frac{5}{13} \\ &\cos \omega_0 \approx 0.917.. \approx 0.92307... \approx \frac{12}{13} \\ &1 = (\frac{12}{13})^2 + (\frac{5}{13})^2 \end{aligned}$$

The duration of the day on Earth is therefore

$$\begin{aligned} d_t &= \arccos - \frac{\frac{60}{169} \cos \lambda \cos^2 t + \sin t \sqrt{\sin^2 \lambda - \frac{25}{169} \cos^2 t}}{\sin \lambda \cdot (1 - \frac{25}{169} \cos^2 t)} \\ &+ \arccos - \frac{\frac{60}{169} \cos \lambda \cos^2 t - \sin t \sqrt{\sin^2 \lambda - \frac{25}{169} \cos^2 t}}{\sin \lambda \cdot (1 - \frac{25}{169} \cos^2 t \sin^2 \omega)} \end{aligned}$$

Graphically the solution space in the case of the Earth looks as follow. Note that the wedge in front has coordinates $(t, \omega\, d) = (\frac{\pi}{2}, \frac{\pi}{2}, 2\pi)$ and the bottom plane has height $d = \pi$.

Plot of z =



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We have on the left the $t - axis$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and on the right the latitude λ of a location with $\omega_0 < \lambda < \frac{\pi}{2}$. The origin of the axis is the on the middle $t - axis$ behind the surface.

The graphic can be continued in the ω direction by axial symmetry around the axis $t = \frac{\pi}{2}$ (and $d = \pi$).

It is interesting to see how sharp the duration on the day decreases after the solstice the closer we are to the polar circle. The duration of 12^h (i.e. $d = \pi$) around the equinox is stable (with slope 0 indeed we have $\frac{\partial}{\partial t} d_{t,\lambda} \,|_{t=\frac{\pi}{2}} = 0$ see next computation, where we introduce three auxiliary functions: N_1 , N_2 and D)

$$
\begin{aligned}
N_1(t) &= \cos\lambda \cos^2 t \cos\omega \sin\omega \\
N_1'(t) \,|_{t=\frac{\pi}{2}} &= 2\cos t \sin t \cos\lambda \cos\omega \sin\omega \,|_{t=\frac{\pi}{2}} = \cos t \cdot \therefore \,|_{t=\frac{\pi}{2}} = 0 \\
N_2(t) &= \sin^2\lambda - \cos^2 t \cdot \sin^2\omega \\
N_2'(t) \,|_{t=\frac{\pi}{2}} &= -2\cos t \sin t \cdot \sin^2\omega = \cos t \cdot \therefore \,|_{t=\frac{\pi}{2}} = 0 \\
D(t) &= \sin\lambda \cdot (1 - \cos^2 t \sin^2\omega)
\end{aligned}
$$

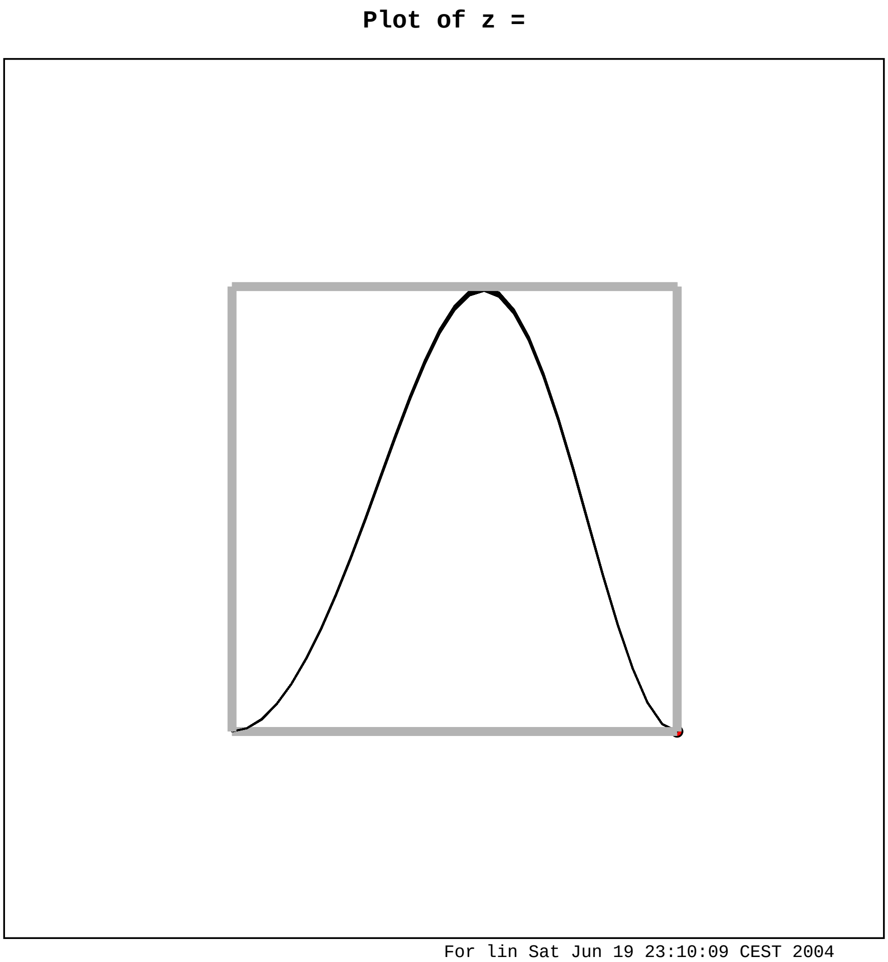
$$
\begin{aligned}
D'(t)\mid_{t=\frac{\pi}{2}} &= -2\cos 2t \sin t \sin^2 \omega) = \cos t \cdot \ddots .. \mid_{t=\frac{\pi}{2}} = 0 \\
d_t &= \arccos - \frac{N_1(t) + \sin t \sqrt{N_2(t)}}{D(t)} \\
\frac{\partial}{\partial t} d_t \mid_{t=\frac{\pi}{2}} &= 0
\end{aligned}
$$

4.2.1 The special case of Berne

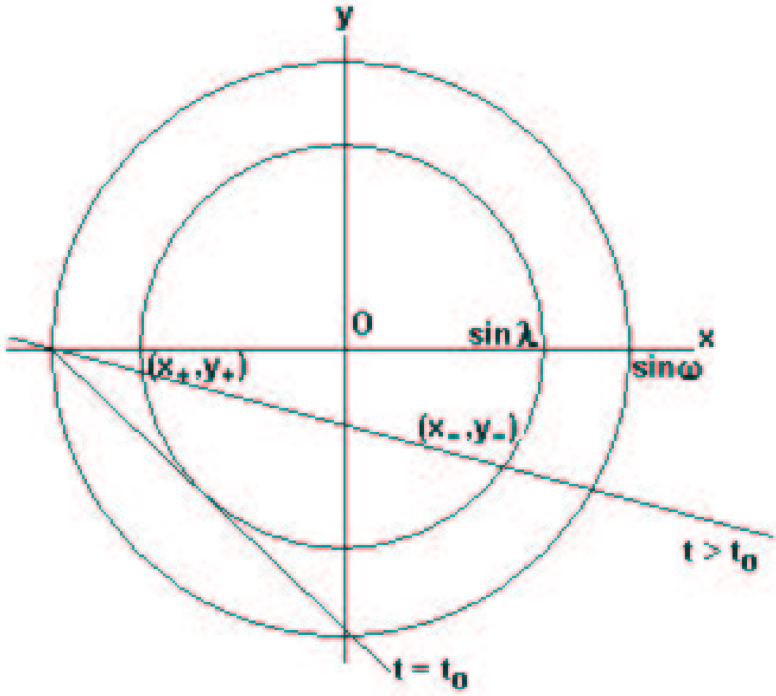
The latitude of Berne is 47^o which corresponds to our latitude $\lambda_0 = \frac{3}{4} rad = 0.75 rad \approx 90^o - 47^o$. The day duration is

$$
\begin{aligned}
d_t &= \arccos - \frac{\frac{60}{169}\cos\frac{3}{4}\cos^2 t + \sin t \sqrt{\sin^2\frac{3}{4} - \frac{25}{169}\cos^2 t}}{\sin\frac{3}{4}\cdot(1 - \frac{25}{169}\cos^2 t)} \\
&+ \arccos - \frac{\frac{60}{169}\cos\frac{3}{4}\cos^2 t - \sin t \sqrt{\sin^2\frac{3}{4} - \frac{25}{169}\cos^2 t}}{\sin\frac{3}{4}\cdot(1 - \frac{25}{169}\cos^2 t \sin^2 \omega)}
\end{aligned}
$$

Which gives us the following graphic



4.3 Above the Polar Circle



We have $0 < \omega, t < \frac{\pi}{2}$ and $0 < \lambda < \omega$. The length of the day is 2π for a period around the Summer solstice, i.e. when below the square root there is a negative or zero value:

$$\begin{aligned} \sin^2\lambda - \cos^2 t \cdot \sin^2\omega &\leq 0 \\ \sin^2\lambda &\leq \cos^2 t \cdot \sin^2\omega \\ \sin\lambda &\leq \cos t \cdot \sin\omega \\ \cos t &\geq \frac{\sin\lambda}{\sin\omega} \\ t &\leq t_0 = \arccos\frac{\sin\lambda}{\sin\omega} \end{aligned}$$

For $\frac{\pi}{2} > t > t_0$ we have a day duration $d_t = 2\pi + \arccos\frac{x-}{\sin\lambda} - \arccos\frac{x+}{\sin\lambda}$.

In summary we have with $t_0 = \arccos\frac{\sin\lambda}{\sin\omega}$

$$\begin{aligned} d_t\,|_{t\leq t_0} &= 2\pi \\ d_t\,|_{t>t_0} &= 2\pi + \arccos - \frac{\cos\lambda\cos^2 t\cos\omega\sin\omega - \sin t\sqrt{\sin^2\lambda - \cos^2 t\cdot\sin^2\omega}}{\sin\lambda\cdot(1-\cos^2 t\sin^2\omega)} \\ &\quad - \arccos - \frac{\cos\lambda\cos^2 t\cos\omega\sin\omega - \sin t\sqrt{\sin^2\lambda - \cos^2 t\cdot\sin^2\omega}}{\sin\lambda\cdot(1-\cos^2 t\sin^2\omega)} \end{aligned}$$

Appendix

The solutions of the general equation where computed with maxima.

```
(C1)solve([x^2+y^2=sin(L)^2,

cos(t)*cos(w)*x+sin(t)*y=-cos(t)*sin(w)*cos(L)],[x,y]);

~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ 2~~~~~~ 2~~~~~~ 2~~~~~~~~ 2~~~~~~ 2~~~~~~ 2

(D1) [[x = - (SIN(t) SQRT(- COS (L) COS (t) SIN (w) + SIN (L) COS (t) COS (w)

~~~~~ 2~~~~~~ 2~~~~~~~~~~~~~~~~ 2

~+ SIN (L) SIN (t)) + COS(L) COS (t) COS(w) SIN(w))

~~~~ 2~~~~~~ 2~~~~~~~~ 2

/(COS (t) COS (w) + SIN (t)), y = (COS(t) COS(w)

~~~~~~~~~~ 2~~~~~~ 2~~~~~~ 2~~~~~~~~ 2~~~~~~ 2~~~~~~ 2~~~~~~~~ 2~~~~~~ 2

~SQRT(- COS (L) COS (t) SIN (w) + SIN (L) COS (t) COS (w) + SIN (L) SIN (t))

~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ 2~~~~~~ 2~~~~~~~~ 2

~- COS(L) COS(t) SIN(t) SIN(w))/(COS (t) COS (w) + SIN (t))],

~~~~~~~~~~~~~~~~~~~~~~ 2~~~~~~ 2~~~~~~ 2~~~~~~~~ 2~~~~~~ 2~~~~~~ 2
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```
[x = (SIN(t) SQRT(- COS (L) COS (t) SIN (w) + SIN (L) COS (t) COS (w)

~~~~~ 2~~~~~~ 2~~~~~~~~~~~~~~~~ 2

~+ SIN (L) SIN (t)) - COS(L) COS (t) COS(w) SIN(w))

~~~~ 2~~~~~~ 2~~~~~~~~ 2

/(COS (t) COS (w) + SIN (t)), y = - (COS(t) COS(w)

~~~~~~~~~~ 2~~~~~~ 2~~~~~~ 2~~~~~~~~ 2~~~~~~ 2~~~~~~ 2~~~~~~~~ 2~~~~~~ 2

~SQRT(- COS (L) COS (t) SIN (w) + SIN (L) COS (t) COS (w) + SIN (L) SIN (t))

~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ 2~~~~~~ 2~~~~~~~~ 2

~+ COS(L) COS(t) SIN(t) SIN(w))/(COS (t) COS (w) + SIN (t))]]as typed
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